VINNITSA NATIONAL AGRARIAN UNIVERSITY

Department of General Engineering Sciences and Labour Safety

CALCULATION OF TRANSIENTS IN ELECTRICAL CIRCUITS OF THE SECOND ORDER

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ALGORITHM FOR CALCULATING TRANSIENTS IN COMPLEX ELECTRICAL CIRCUITS

1. Compose a system of equations for an electric circuit according to Kirchhoff's rules in an instantaneous form.
2. Based on the system of equations to obtain an inhomogeneous differential equation.
3. Based on the inhomogeneous differential equation to obtain the characteristic equation. Find its solution.
4. Find homogeneous solution (own component).
5. Find particular solution (forced component).

EXAMPLE 1 OF CALCULATING THE TRANSIENT PROCESS IN A BRANCHED CIRCLE OF THE SECOND ORDER

Step1: $i_S = i_C = i_L \rightarrow \frac{v_R}{R_T} = i_L + i_C$

$$-v_S + v_R + v_L + v_C = 0 \rightarrow v_R + v_L + v_C = v_S$$

$$i_L R + L\frac{di_L}{dt} + v_C(t=0) + \int_0^t \frac{i_L(t')}{C}dt' = v_S \rightarrow L\frac{d^2 i_L}{dt^2} + R\frac{di_L}{dt} + \frac{i_L}{C} = \frac{dv_S}{dt} = 0$$

Step2: $v_C(t=0^-) = 5\text{ V} = v_C(t=0^+), i_L(t=0^-) = 0\text{ A} = i_L(t=0^+)$

$$i_L(t=0^+)R + L\frac{di_L}{dt}(t=0^+) + v_C(t=0) = v_S \rightarrow 1\frac{di_L}{dt}(t=0^+) + 5\text{ V} = 25\text{V} \rightarrow \frac{di_L}{dt}(t=0^+) = 20\text{A/s}$$

Step3: $L\frac{d^2 i_L}{dt^2} + R\frac{di_L}{dt} + \frac{i_L}{C} = 0 \rightarrow LC\frac{d^2 i_L}{dt^2} + RC\frac{di_L}{dt} + i_L = 0 : \frac{1}{\omega_n^2}\frac{d^2 x(t)}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx(t)}{dt} + x(t) = K_S f(t)$

$$\frac{1}{\omega_n^2} = LC \rightarrow \omega_n = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{10^{-6}}} = 1000\,(\text{rad/s}), \frac{2\zeta}{\omega_n} = RC \rightarrow \zeta = \frac{RC\omega_n}{2} = \frac{R}{2}\sqrt{\frac{C}{L}} = \frac{5000}{2}\sqrt{\frac{10^{-6}}{1}} = 2.5$$

$\rightarrow$ Overdamped response

$i_L(t) = \alpha_1 e^{s_1 t} + \alpha_2 e^{s_2 t}$ where $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

Complete Response (forced response = 0)

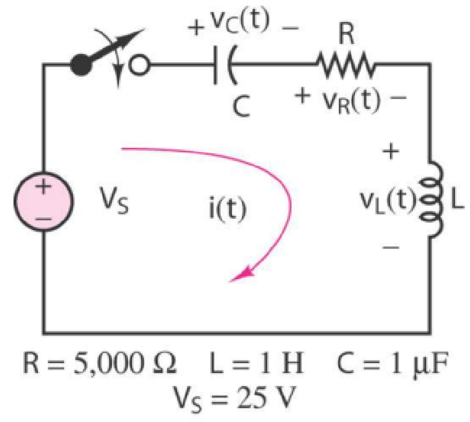
$$i_L(t) = \alpha_1 e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t} + \alpha_2 e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t}$$

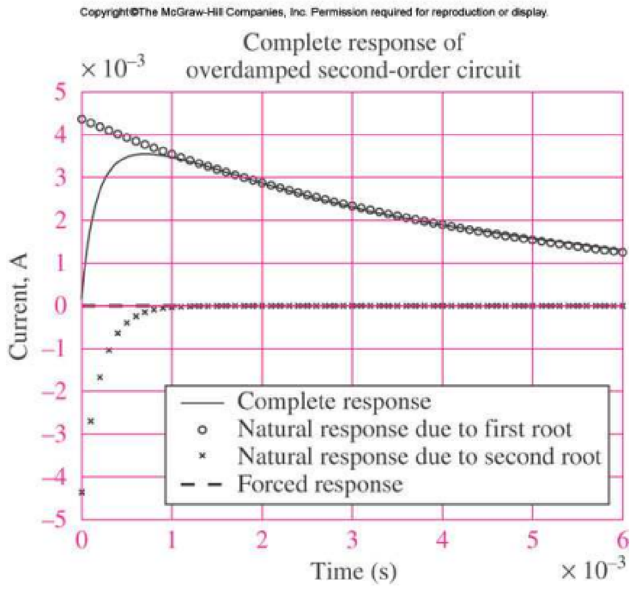
Step4: Using $0\text{ A} = i_L(t=0^+)$ and $\frac{di_L}{dt}(t=0^+) = 20\text{A/s}$, determine the constants α_1 and α_2

$$i_L(t=0^+) = 0 = \alpha_1 + \alpha_2$$

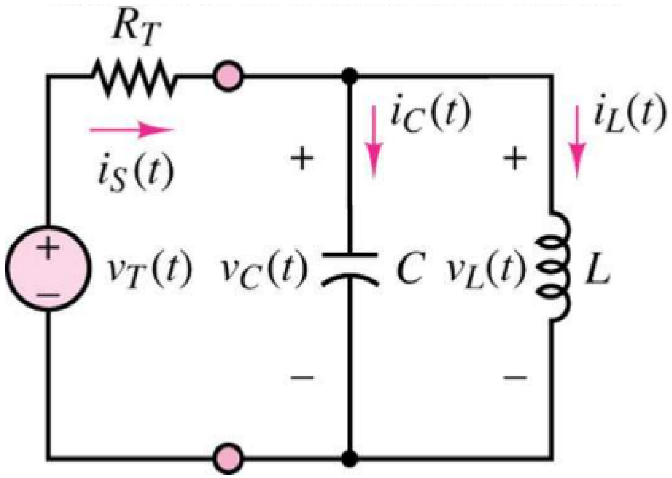
$$\frac{di_L}{dt} = \alpha_1\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t} + \alpha_2\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t}$$

$$\frac{di_L}{dt}(t=0^+) = 20 = \alpha_1\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right) + \alpha_2\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)$$





EXAMPLE 2 OF CALCULATING THE TRANSIENT PROCESS IN A BRANCHED CIRCLE OF THE SECOND ORDER



$$i_S = i_C + i_L \rightarrow \frac{v_R}{R_T} = i_L + i_C$$

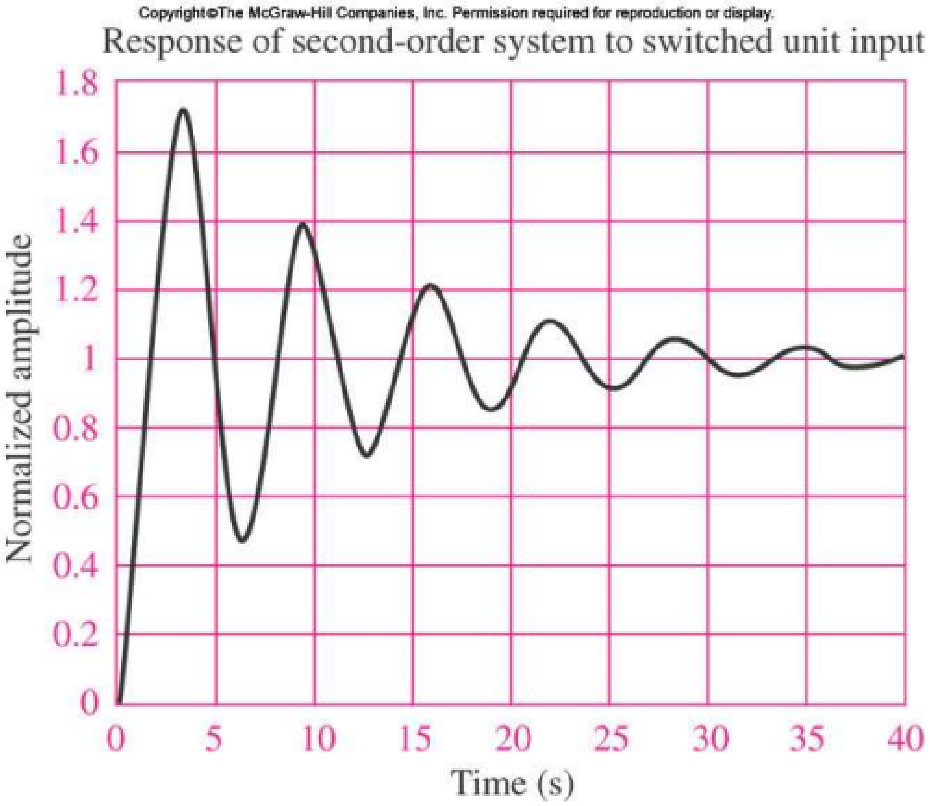
$$-v_T + v_R + v_L = 0 \rightarrow v_R = v_T - v_L \text{ and } -v_T + v_R + v_C = 0 \rightarrow v_R = v_T - v_C$$

$$\frac{v_R}{R_T} = i_L + i_C \rightarrow \frac{1}{R_T}\left(v_T - L\frac{di_L}{dt}\right) = i_L + C\frac{dv_C}{dt} = i_L + C\frac{d}{dt}\left(L\frac{di_L}{dt}\right)$$

$$\frac{1}{R_T}\left(v_T - L\frac{di_L}{dt}\right) = i_L + LC\frac{d^2 i_L}{dt^2} \rightarrow \frac{v_T}{R_T} = LC\frac{d^2 i_L}{dt^2} + \frac{L}{R_T}\frac{di_L}{dt} + i_L$$

$$a_2 \frac{d^2x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = b_0 f(t) \rightarrow \frac{1}{\omega_n^2} \frac{d^2x(t)}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx(t)}{dt} + x(t) = K_S f(t)$$

where the constants $\omega_n = \sqrt{a_0/a_2}, \zeta = (a_1/2)\sqrt{1/a_0 a_2}$ and $K_S = b_0/a_0$ termed the natural frequency, the damping ratio, and the DC gain, respectively.



$\omega_n = 1, \zeta = 0.1$ and $K_S = 1$

- The final value of 1 is predicted by the DC gain K_S=1, which tells us about the steady state.
- The period of oscillation of the response is related to the natural frequency w_n=1 leads to T=2 pi/w_n = 6.28 sec.
- The reduction in amplitude of the oscillation is governed by the damping ratio. With large damping ratio, the response not overshoots (oscillates) but looks like the first order response.
- Damping -> friction effect

$$\frac{1}{\omega_n^2}\frac{d^2x(t)}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx(t)}{dt}+x(t)=K_S f(t)$$

Natural Response

$$\frac{1}{\omega_n^2}\frac{d^2x_N(t)}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx_N(t)}{dt}+x_N(t)=0$$

$x_N(t)=\alpha_1 e^{s_1 t}+\alpha_2 e^{s_2 t}$ where $s_{1,2}=-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$

Case 1 : Real and distinct roots. $(\zeta>1) \rightarrow$ Overdamped response

$\rightarrow$ Look like the first order system

$s_{1,2}=-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$

Case 2 : Real and repeated roots. $(\zeta=1)$

$\rightarrow$ Critically overdamped response $\rightarrow$ Oscillation

$s_{1,2}=-\omega_n$

Case 3 : Complex roots. $(\zeta<1) \rightarrow$ Underdamped response $\rightarrow$ Oscillation

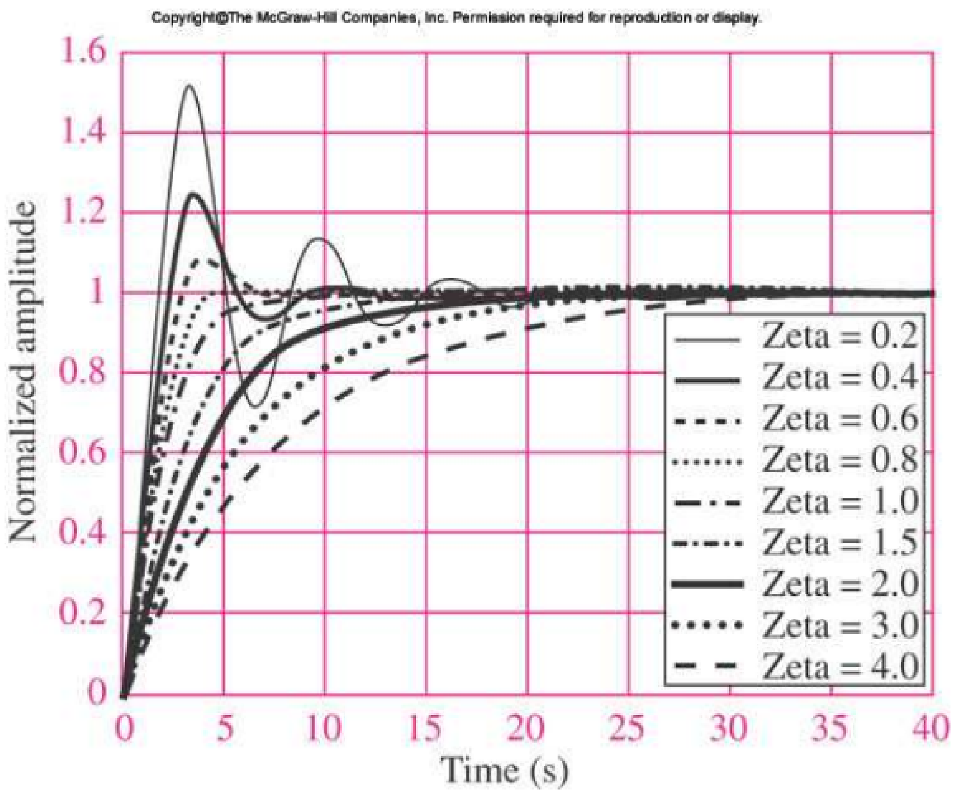
$s_{1,2}=-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

Forced Response due to DC (where $f(t)=F$): $\frac{dx_F(t)}{dt}\rightarrow 0$

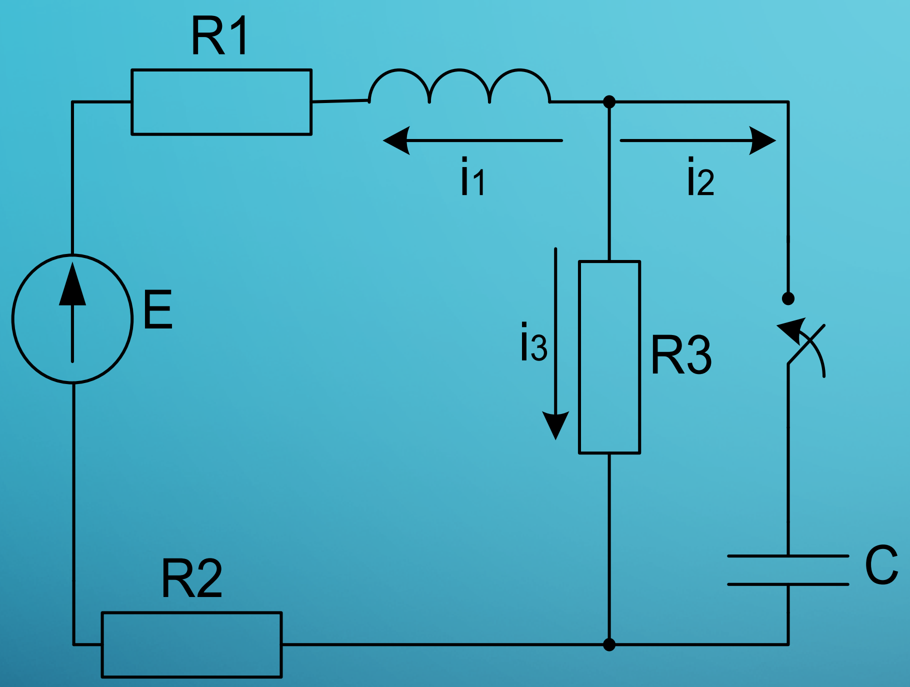
$$\frac{1}{\omega_n^2}\frac{d^2x_F(t)}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx_F(t)}{dt}+x_F(t)=K_S f(t)\ \ t\ge 0 \rightarrow x_F(t)=K_S F\ \ t\ge 0$$

Complete Response

$x(t)=x_N(t)+x_F(t)$ α_1 and α_2 is constants that will be determined by the initial conditions.



EXAMPLE OF CALCULATING THE TRANSIENT PROCESS IN A BRANCHED CIRCLE OF THE SECOND ORDER (WITH NUMERICAL VALUES)



U=50 V

R1=10 Ω

R2=90 Ω

R3=100 Ω

L=10 mH

C=10 μF

$E := 50 \quad R1 := 10 \quad R2 := 90$

$R3 := 100 \quad L := 10 \cdot 10^{-3} \quad C := 10 \cdot 10^{-6}$

Initial values

$UC0 := 0$

$$I10 := \frac{-E}{R1 + R2 + R3} = -0.25$$

System of equations according to Kirchhoff's rules

$$i1(t) + i2(t) + i3(t) = 0$$

$$-i1(t) \cdot (R1 + R2) - L \cdot \frac{di(1)}{d(t)} + i(3) \cdot R3 = E$$

$$-i1(t) \cdot (R1 + R2) - L \cdot \frac{di(1)}{d(t)} + UC(t) = E$$

$$i2(t) = C \cdot \frac{dUC(t)}{d(t)}$$

Characteristic equation

Given

$$R1 + p\cdot L + \frac{R3\cdot\frac{1}{p\cdot C}}{R3 + \frac{1}{p\cdot C}} + R2 = 0$$

$$\mathrm{Find}(p) \rightarrow \left(500\cdot\sqrt{41} - 5500 \quad -500\cdot\sqrt{41} - 5500\right)$$

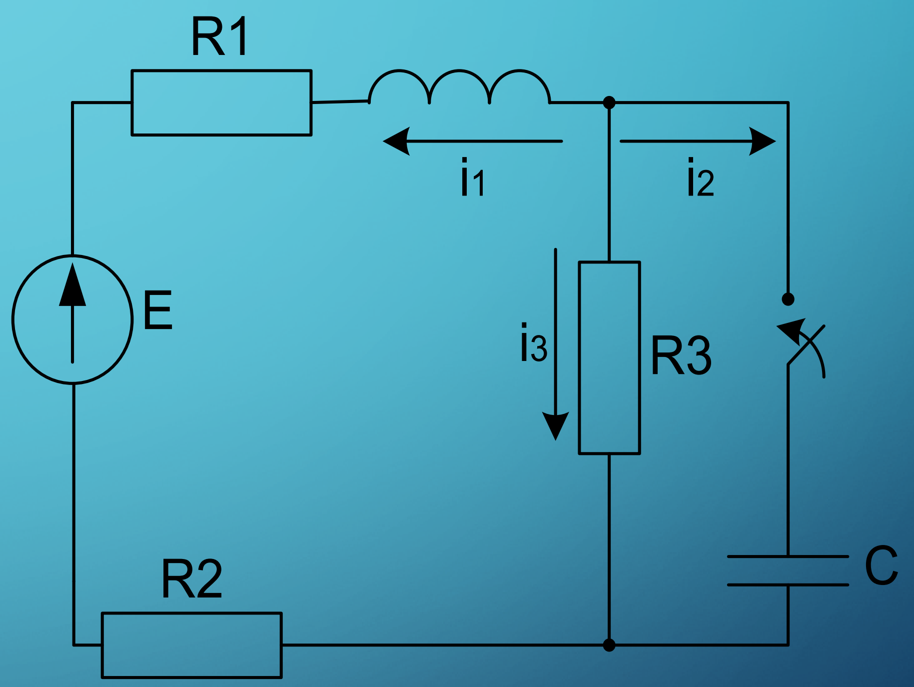
$$p1 := 500\cdot\sqrt{41} - 5500 = -2.298\times 10^{3}$$

$$p2 := -500\cdot\sqrt{41} - 5500 = -8.702\times 10^{3}$$

Own component of capacitor voltage

$$UCv(t) = A1\cdot e^{p1\cdot t} + A2\cdot e^{p2\cdot t}$$

$$UCv(t) = A1\cdot e^{-2.298\times 10^{3}\cdot t} + A2\cdot e^{-8.702\times 10^{3}\cdot t}$$



Forced component of capacitor voltage

$$UCpr := \frac{E}{R1 + R2 + R3} \cdot R3 = 25$$

Загальний розв'язок напруги на ємності

$$UC(t) = UCv(t) + UCpr$$

$$UC(t) = A1 \cdot e^{-2.298\times10^3 \cdot t} + A2 \cdot e^{-8.702\times10^3 \cdot t} + 25$$

$$\frac{dUC(t)}{dt} = -2.298 \times 10^3 \cdot A1 \cdot e^{-2.298\times10^3 \cdot t} + -8.702 \times 10^3 \cdot A2 \cdot e^{-8.702\times10^3 \cdot t}$$

Substitute t=0

$$\frac{dUC(0)}{dt} = -2.298 \times 10^3 \cdot A1 - 8.702 \times 10^3 \cdot A2$$

$$UC0 = A1 \cdot e^{-2.298\times10^3 \cdot 0} + A2 \cdot e^{-8.702\times10^3 \cdot 0} + 25$$

According to initial values

Given

$$0 = A1 + A2 + 25$$

$$10 \cdot 10^{-6} \cdot \left(-2.298 \times 10^3 \cdot A1 - 8.702 \times 10^3 \cdot A2\right) = 0.25 + \frac{A1}{100} + \frac{A2}{100} + 0.25$$

$$Find(A1, A2) \rightarrow \begin{pmatrix} -30.067145534041224235 \\ 5.0671455340412242349 \end{pmatrix}$$

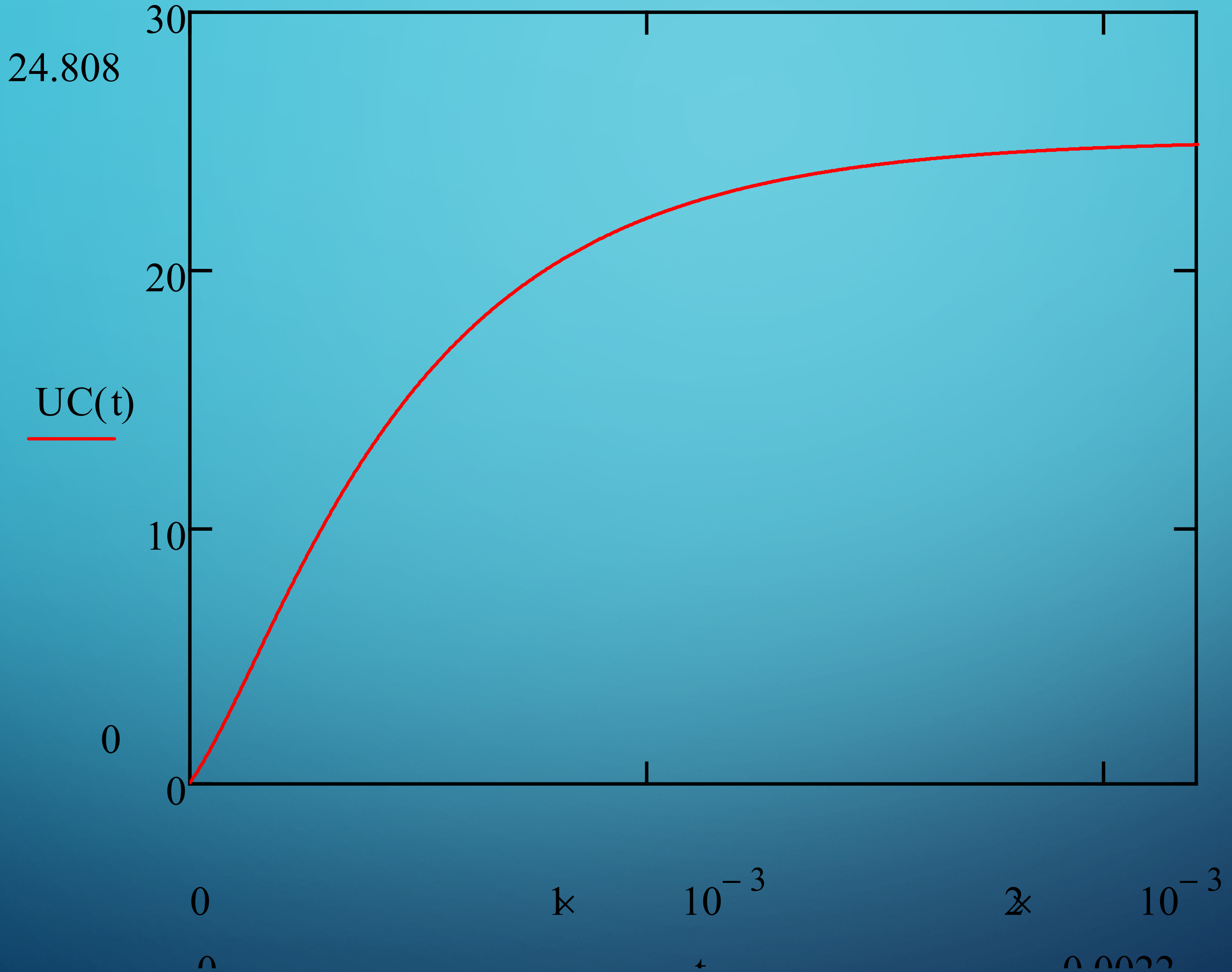
General solution of capacitor voltage

$$UC(t) := -30.067 \cdot e^{-2.298 \times 10^{3} \cdot t} + 5.067 \cdot e^{-8.702 \times 10^{3} \cdot t} + 25$$

$$\tau 1 := \frac{1}{2.298 \times 10^{3}} = 4.352 \times 10^{-4}$$

$$\tau 2 := \frac{1}{8.702 \times 10^{3}} = 1.149 \times 10^{-4}$$

$$tpp := 5 \cdot \tau 1 = 2.176 \times 10^{-3}$$


30
24.808
20
UC(t)
10
0
0
0
1 × 10^−3
2 × 10^−3